APPROXIMATE SOLUTIONS OF A NON-LINEAR DIFFERENTIAL EQUATION USING LAPLACE-TRANSFORM AND REVERSION-OF-SERIES TECHNIQUES

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THESIS

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ABSTRACT

The reversion-of-series method is extended to the s - domain by utilizing non-linear Laplace transforms. The reversion of series in the s - domain is applied to a non-linear differential equation and approximate solutions are obtained. The approximate solution is modified for the case where the steady state is a constant value by calculating the exact steady-state value and applying it to the reversion approximation. The non-linear differential equation considered is Duffing's equation with a damping term and sinusoidal and constant forcing functions. The theoretical solutions are compared to machine solutions.

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I. INTRODUCTION

Utilization of the Laplace transformation to obtain an accurate analytical solution of a non-linear differential equation has been prevented, in past, by the inability to obtain an appropriate expression for the Laplace transform of the non-linear terms. Baycura [Ref. 1] developed an expression for the Laplace transform of certain non-linear terms and Brady [Ref. 2] obtained a formulation for the general case. The non-linear transform is used in conjunction with the reversion-of-series method to obtain an approximate solution for a non-linear differential equation.

The approximate solution is formulated and examined in detail by comparison to a machine solution. A solution obtained by modifying the approximate solution is also derived and examined in detail.

II. TRANSFORM EXPRESSION FOR INTEGRAL POWERS OF A FUNCTION

The general formula for the Laplace transform of a function raised to an integral power [Ref. 3] is:

$$\boxed{\mathcal{L}[x^n(t)] = s^{n-1}X^n(s)} \qquad (1)$$

where n is a positive integer greater than zero.

III. REVERSION OF SERIES

If a series is represented by

$$x = a_1y + a_2y^2 + a_3y^3 + a_4y^4 + a_5y^5 + \dots \quad (a_1 \neq 0) \tag{2}$$

the coefficients of the series

$$y = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots \tag{3}$$

are given by [Ref. 4]:

$$A_1 = \frac{1}{a_1} \qquad A_2 = -\frac{a_2}{a_1^{\,3}} \qquad A_3 = \frac{1}{a_1^{\,5}}(2a_2^{\,2} - a_1a_3)$$

$$A_4 = \frac{1}{a_1^{\,7}}(5a_1a_2a_3 - a_1^{\,2}a_4 - 5a_2^{\,3})$$

$$A_5 = \frac{1}{a_1^{\,9}}(6a_1^{\,2}a_2a_4 + 3a_1^{\,2}a_3^{\,2} + 14a_2^{\,4} - a_1^{\,3}a_5 - 21a_1a_2^{\,2}a_3)$$

Utilizing the relationship expressed in Equation (1) the Laplace transform of Equation (2) becomes:

$$X(s) = a_1Y(s) + a_2sY^2(s) + a_3s^2Y^3(s) + a_4s^3Y^4(s) + \dots \tag{4}$$

Let:

$$b_1 = a_1 \qquad b_2 = a_2s \qquad b_3 = a_3s^2 \qquad b_4 = a_4s^3 \qquad \dots$$

Equation (4) can then be rewritten as:

$$X(s) = b_1Y(s) + b_2Y^2(s) + b_3Y^3(s) + b_4Y^4(s) \dots \tag{5}$$

The Laplace transform of Equation (3) is:

$$Y(s) = A_1X(s) + A_2sX^2(s) + A_3s^2X^3(s) + A_4s^3X^4(s) \quad \ldots \qquad (6)$$

Let:

$$B_1 = a_1 \quad B_2 = A_2s \quad B_3 = A_3s^2 \quad B_4 = A_4s^3 \quad \ldots$$

Equation (6) can then be rewritten as:

$$Y(s) = B_1X(s) + B_2X^2(s) + B_3X^3(s) + B_4X^4(s) \ldots \qquad (7)$$

The coefficients of Equation (7) can be expressed in terms of the coefficients of Equation (5):

$$B_1 = \frac{1}{b_1} \quad B_2 = -\frac{b_2}{b_1^3} \quad B_3 = \frac{1}{b_1^5}(2b_2^2 - b_1b_3)$$

$$B_4 = \frac{1}{b_1^7}(5b_1b_2b_3 - b_1^2b_4 - 5b_2^3)$$

$$B_5 = \frac{1}{b_1^9}(6b_1^2b_2b_4 + 3b_1^2b_3^2 + 14b_2^4 - b_1^3b_5 - 21b_1b_2^2b_3)$$

Thus, Y(s) can be expressed as a function of X(s) by the reversion of Equation (5) into Equation (7).

IV. GENERAL SOLUTION OF A NON-LINEAR DIFFERENTIAL EQUATION

The technique illustrated in the previous section will now be used to solve Duffing's equation with a damping term and initial conditions. The equation is expressed as:

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \omega_0^2 y + hy^3 = x(t) \tag{8}$$

where α, ω_0^2 and h are constants and x(t) is a forcing function. The Laplace transform of Equation (8) is:

$$(s^2 + \alpha s + \omega_0^2)Y(s) + hs^2Y^3(s) = X(s) + sy(0) + \left.\frac{dy}{dt}\right|_0 + \alpha y(0) \tag{9}$$

Let:

$$b_1 = s^2 + \alpha s + \omega_0^2$$

$$b_3 = hs^2$$

$$Z(s) = X(s) + y(0)(s+\alpha) + \left.\frac{dy}{dt}\right|_0$$

Equation (9) can then be written as:

$$b_1Y(s) + b_3Y^3(s) = Z(s) \tag{10}$$

Consider Equation (10) as series with b_n equal to zero for all n except n equal to one and n equal to three.

By reversion of series:

$$Y(s) = \frac{Z(s)}{(s^2+\alpha s+\omega_0^2)} - \frac{hs^2Z^3(s)}{(s^2+\alpha s+\omega_0^2)^4} + \frac{3h^2s^4Z^5(s)}{(s^2+\alpha s+\omega_0^2)^7} + \ldots \tag{11}$$

The denominator of each term in Equation (11) can be factored. Thus:

$$s^2 + \alpha s + \omega_0^2 = \left[s + \frac{\alpha}{2} + \frac{(\alpha^2-4\omega_0^2)^{\frac{1}{2}}}{2}\right]\left[s + \frac{\alpha}{2} - \frac{(\alpha^2-4\omega_0^2)^{\frac{1}{2}}}{2}\right] \tag{12}$$

Let:

$$A = \frac{\alpha}{2} + \frac{1}{2}(\alpha^2-4\omega_0^2)^{\frac{1}{2}} \tag{13}$$

$$B = \frac{\alpha}{2} - \frac{1}{2}(\alpha^2-4_0^2)^{\frac{1}{2}} \tag{14}$$

Equation (11) can then be rewritten as:

$$Y(s) = \frac{Z(s)}{(s+A)(s+B)} - \frac{hs^2Z^3(s)}{(s+A)^4(s+B)^4} + \frac{3h^2s^4Z^5(s)}{(s+A)^7(s+B)^7} + \ldots \tag{15}$$

Equation (15) represents the general solution in the s - domain. Inspection of Equation (15) reveals that the form of the time solution depends on the nature of the forcing function and the relationship between α^2 and $4\omega_0^2$. Further, if the forcing function is Laplace transformable and if h is equal to zero, the solution is exact.

V. APPLICATION OF THE GENERAL SOLUTION

The general solution just derived will now be applied for zero initial conditions with constant and sinusoidal forcing functions. The time solutions are given in terms of numbered coefficients. The formula for each coefficient is listed in the Appendix.

A. CONSTANT FORCING FUNCTION

The inverse Laplace transform of Equation (15) yields three different solutions depending on the nature of the radical in Equations (13) and (14).

1. Radical Equal to Zero

For the radical to be equal to zero, then:

$$\alpha^2 = 4\omega_0^2 \tag{16}$$

and

$$A = B = \frac{\alpha}{2} \tag{17}$$

The general solution in the s - domain is then:

$$Y(s) = \frac{c}{s(s+A)^2} - \frac{hc^3}{s(s+A)^8} + \frac{3h^2c^5}{s(s+A)^{14}} + \ldots \tag{18}$$

where c is the constant forcing function.

The inverse Laplace transform of the first three terms of Equation (18) yields:

$$y(t) = K_1 + (K_2+K_3t)e^{-At} + K_4 + (K_5+K_6t+K_7t^2 + \ldots + K_{18}t^{13})e^{-At} \tag{19}$$

where the K_i, i = 1, 2, 3, ..., are defined in the Appendix.

An examination of Equation (19) indicates that the initial conditions are satisfied. The effect of the various parameters on the solution can be seen by examining the steady-state solution which can be expressed as:

$$y(t) = \frac{c}{\omega_0^2}\left[1 - \frac{hc^2}{(\omega_0^2)^3} + 3\left[\frac{hc^2}{(\omega_0^2)^3}\right]^2 -12\left[\frac{hc^2}{(\omega_0^2)^3}\right]^3 + \ldots\right] \tag{20}$$

It should be noted that only the first three terms of Equation (20) are utilized in Equation (19). Since Equation (20) is an alternating infinite series, a condition for convergence [Ref. 5] is that the magnitude of each successive term must decrease. This implies that the applicability of the reversion method is limited in the steady state to the situation where hc^2 is small compared to $(\omega_0^2)^3$.

Using the arbitrary values c = 1.0, α = 4.0, ω_0^2 = 4.0 and h = 1.0, the solution represented by Equation (19) is compared in Table I to a machine solution which was obtained using Runge-Kutta Adams - Moulton with error check. As the results indicate, a relatively accurate approximate solution was obtained. In Table II the solutions are compared for the coefficient of the non-linear term equal to ten. In this case the solution was a good approximation, but not as accurate as in the previous example. Examination

TABLE I

$\alpha=4.0,\ \omega_0^2=4.0,\ h=1.0,\ c=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0661	0.0661	0.0661	0.0000
1.0	0.1485	0.1485	0.1484	0.0001
1.5	0.2002	0.2002	0.1997	0.0005
2.0	0.2271	0.2269	0.2259	0.0010
2.5	0.2399	0.2394	0.2379	0.0015
3.0	0.2457	0.2447	0.2430	0.0017
3.5	0.2482	0.2466	0.2451	0.0015
4.0	0.2492	0.2471	0.2458	0.0013
4.5	0.2497	0.2471	0.2461	0.0010
5.0	0.2499	0.2467	0.2462	0.0005
5.5	0.2499	0.2466	0.2463	0.0003
6.0	0.2500	0.2465	0.2463	0.0002
6.5	0.2500	0.2464	0.2463	0.0001
7.0	0.2500	0.2463	0.2463	0.0000
7.5	0.2500	0.2463	0.2463	0.0000

TABLE II

$\alpha=4.0,\ \omega_0^2=4.0,\ h=10.0,\ c=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0661	0.0661	0.0660	0.0001
1.0	0.1485	0.1485	0.1477	0.0008
1.5	0.2002	0.1997	0.1956	0.0031
2.0	0.2271	0.2251	0.2160	0.0091
2.5	0.2399	0.2347	0.2221	0.0126
3.0	0.2457	0.2357	0.2232	0.0125
3.5	0.2482	0.2327	0.2229	0.0098
4.0	0.2492	0.2285	0.2227	0.0058
4.5	0.2497	0.2246	0.2225	0.0021
5.0	0.2499	0.2219	0.2225	-0.0006
5.5	0.2499	0.2205	0.2225	-0.0020
6.0	0.2500	0.2202	0.2225	-0.0023
6.5	0.2500	0.2206	0.2225	-0.0019
7.0	0.2500	0.2220	0.2225	-0.0005
7.5	0.2500	0.2233	0.2225	0.0008
8.0	0.2500	0.2246	0.2225	0.0021
9.0	0.2500	0.2268	0.2225	0.0043
15.0	0.2500	0.2292	0.2225	0.0067

of Equation (20) reveals that for h equal to ten, successive terms decrease in magnitude, but not as rapidly as in the case h was equal to one. If four terms of Equation (20) had been utilized in the solution, a steady-state value of 0.2178 would have been obtained resulting in a better approximation. Table III represents a comparison of the solutions for the forcing function equal to ten. In this case the solution limits were exceeded and a large positive number was obtained for the steady state. If an additional term of Equation (20) had been included in the solution, a larger negative value would have been obtained for the steady state. Good results were realized however, for the initial portion of the transient solution, indicating the applicability of the reversion method in this region. The linear solution represented by the first two terms of Equation (19) was also included in the tables to illustrate a less accurate but simplified approximate solution.

2. Radical Greater than Zero

The relationship between α^2 and $4\omega_0^2$ is expressed as:

$$\alpha^2 > 4\omega_0^2 \ . \tag{21}$$

Equation (15) is then expressed as:

$$Y(s) = \frac{c}{s(s+A)(s+B)} - \frac{hc^3}{s(s+A)^4(s+B)^4} + \frac{3h^2c^5}{s(s+A)^7(s+B)^7} + \ldots \tag{22}$$

The inverse Laplace transform of the first two terms of Equation (22) yields:

TABLE III

$\alpha=4.0,\ \omega_0^2=4.0,\ h=1.0,\ c=10.0$

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.2	0.1539	0.1539	0.1539	0.0000
0.4	0.4780	0.4780	0.4770	0.0010
0.6	0.8434	0.8433	0.8361	0.0072
0.8	1.1877	1.1867	1.1628	0.0239
1.0	1.4850	1.4807	1.4085	0.0722
1.5	2.0020	1.9560	1.6365	0.3195
2.0	2.2710	2.0727	1.5880	0.4847
2.5	2.3990	1.8907	1.5006	0.3901
3.0	2.4570	1.5230	1.5515	-0.0285
3.5	2.4818	1.1488	1.5567	-0.4079
4.0	2.4925	0.9814	1.5575	-0.5761
4.5	2.4969	1.2081	1.5569	-0.3488
5.0	2.4988	1.9344	1.5567	0.3777
6.0	2.4998	4.7740	1.5568	3.2172
7.0	2.5000	8.5235	1.5568	6.9967
8.0	2.5000	11.9169	1.5568	10.3601
9.0	2.5000	14.3046	1.5568	12.7478
10.0	2.5000	15.6965	1.5568	14.1397
20.0	2.5000	16.9043	1.5568	15.3475

$$y(t) = K_{20} + K_{21}e^{-At} + K_{22}e^{-Bt} + K_{23} + (K_{24} + K_{25}t + K_{26}t^2 + K_{27}t^3)e^{-At}$$

$$+ (K_{28} + K_{29}t + K_{30}t^2 + K_{31}t^3)e^{-Bt} \quad . \tag{23}$$

The inverse Laplace transform of additional terms of Equation (22) results in a steady-state solution given by Equation (20). The region in which the solution can be applied is then the same as discussed in the previous section. Using the values $c = 1.0$, $\alpha = 6.0$, $\omega_0^2 = 4.0$ and $h = 1.0$, the solution represented by Equation (23) and the solution obtained considering the first three terms of Equation (23) are compared to a machine solution in Table IV. The small errors listed indicate that a highly accurate approximation was obtained. Table V illustrates the effect of increasing the coefficient of the non-linear term to ten. A less accurate approximation was obtained. The effect of increasing the magnitude of the forcing function was not illustrated; however for the parameter values used in Table IV if c had been increased to ten, a steady-state value of -1.4063 would have been obtained indicating that the solution limits would have been exceeded. The linear solution is seen to be a less accurate approximation.

3. Radical Imaginary

The relationship between α^2 and $4\omega_0^2$ is:

$$4\omega_0^2 > \alpha^2 \quad . \tag{24}$$

TABLE IV

$\alpha=6.0,\ \omega_0^2=4.0,\ h=1.0,\ c=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0533	0.0533	0.0533	0.0000
1.0	0.1139	0.1139	0.1138	0.0001
1.5	0.1570	0.1569	0.1568	0.0001
2.0	0.1865	0.1864	0.1860	0.0004
2.5	0.2066	0.2064	0.2058	0.0006
3.0	0.2204	0.2200	0.2191	0.0009
3.5	0.2298	0.2292	0.2281	0.0011
4.0	0.2362	0.2353	0.2341	0.0012
5.0	0.2436	0.2420	0.2408	0.0012
6.0	0.2470	0.2448	0.2438	0.0010
7.0	0.2486	0.2459	0.2452	0.0007
8.0	0.2494	0.2462	0.2459	0.0003
9.0	0.2497	0.2463	0.2460	0.0003
10.0	0.2499	0.2463	0.2462	0.0001
11.0	0.2499	0.2462	0.2462	0.0000
12.0	0.2500	0.2462	0.2463	-0.0001
13.0	0.2500	0.2461	0.2463	-0.0002

TABLE V

$\alpha=6.0$, $\omega_0^2=4.0$, $h=10.0$, $c=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0533	0.0533	0.0533	0.0000
1.0	0.1139	0.1139	0.1136	0.0003
1.5	0.1570	0.1568	0.1553	0.0015
2.0	0.1865	0.1857	0.1821	0.0036
2.5	0.2066	0.2047	0.1986	0.0061
3.0	0.2204	0.2165	0.2086	0.0079
3.5	0.2298	0.2234	0.2145	0.0089
4.0	0.2362	0.2269	0.2179	0.0090
4.5	0.2406	0.2281	0.2196	0.0085
5.0	0.2436	0.2278	0.2210	0.0068
5.5	0.2465	0.2266	0.2216	0.0050
6.0	0.2470	0.2250	0.2220	0.0030
6.5	0.2480	0.2231	0.2222	0.0009
7.0	0.2486	0.2213	0.2223	-0.0010
8.0	0.2494	0.2181	0.2224	-0.0033
9.0	0.2497	0.2157	0.2225	-0.0068
10.0	0.2499	0.2139	0.2225	-0.0086
12.0	0.2500	0.2120	0.2225	-0.0109
15.0	0.2500	0.2110	0.2225	-0.0115

Let:

$$\frac{1}{2}\left(\alpha^2-4\omega_0^2\right)^{\frac{1}{2}} = jD.$$

Then:

$$A = \frac{\alpha}{2} + jD \tag{25}$$

and

$$B = \frac{\alpha}{2} - jD\ . \tag{26}$$

Equation (15) is then expressed as:

$$Y(s) = \frac{c}{s(s+A)(s+B)} - \frac{hc^3}{s(s+A)^4(s+B)^4} + \frac{3h^2c^3}{s(s+A)^7(s+B)^7} + \ldots \tag{27}$$

The inverse Laplace transform of the first two terms of Equation (27) yields:

$$y(t) = K_{40} + K_{41}e^{-At} + K_{42}e^{-Bt} + K_{43} + (K_{44} + K_{45}t + K_{46}t^2 + K_{47}t^3)e^{-At}$$
$$+ (K_{48} + K_{49}t + K_{50}t^2 + K_{51}t^3)e^{-Bt}. \tag{28}$$

Since A and B are complex, the coefficients of the time-dependent portion of Equation (28) are complex. Equation (28) can then be rewritten as:

$$y(t) = K_{40} + (M_1+jM_2)e^{-At} + (M_1-jM_2)e^{-Bt} + K_{43}$$
$$+ [(M_3+jM_4) + (M_5+jM_6)t + (M_7+jM_8)t^2 + (M_9+jM_{10})t^3]e^{-At}$$
$$+ [(M_3-jM_4) + (M_5-jM_6)t + (M_7-jM_8)t^2 + (M_9-jM_{10})t^3]e^{-Bt}. \tag{29}$$

Combining terms Equation (29) becomes:

$$y(t) = K_{40} + 2M_1 e^{-\frac{\alpha}{2}t}\cos(Dt) + 2M_2 e^{-\frac{\alpha}{2}t}\sin(Dt)$$

$$+ (2M_3+2M_5t+2M_7t^2+2M_9t^3)e^{-\frac{\alpha}{2}t}\cos(Dt)$$

$$+ (2M_4+2M_6t+2M_8t^2+2M_{10}t^3)e^{-\frac{\alpha}{2}t}\sin(Dt) \ . \tag{30}$$

In this case as in the previous two cases considered, the steady-state solution is given by Equation (20). In Table VI, the reversion solution is compared to a machine solution for $\alpha = 1.0$, $\omega_0^2 = 4.0$, $h = 1.0$ and $c = 1.0$. In Table VII, the solutions are compared for the non-linear coefficient equal to ten. A comparison of the results listed in Tables VI and VII with the results obtained in the previous cases for the same values of ω_0^2, h and c, indicates that the reversion method yields a less accurate transient solution when the damping coefficient is small. In this case the linear solution is seen to be a better approximation over a portion of the transient solution.

B. SINUSOIDAL FORCING FUNCTION

Substitution of a constant-amplitude sinusoidal forcing function, $c\sin(\omega t)$, into the first two terms of Equation (15) and utilization of the trigonometric identity [Ref. 6]

$$\sin^3(\omega t) = \frac{3}{4}\sin(\omega t) - \frac{1}{4}\sin(3\omega t) \tag{31}$$

yields:

TABLE VI

$\alpha=1.0,\ \omega_0^2=4.0,\ h=1.0,\ c=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0982	0.0982	0.0982	0.0000
1.0	0.2677	0.2677	0.2672	0.0005
1.5	0.3576	0.3574	0.3534	0.0040
2.0	0.3343	0.3330	0.3235	0.0095
2.5	0.2591	0.2550	0.2463	0.0087
3.0	0.2069	0.1985	0.1994	-0.0009
3.5	0.2064	0.1951	0.2060	-0.0109
4.0	0.2377	0.2279	0.2402	-0.0123
4.5	0.2655	0.2620	0.2655	-0.0035
5.0	0.2712	0.2743	0.2663	0.0080
5.5	0.2593	0.2642	0.2516	0.0126
6.0	0.2453	0.2453	0.2387	0.0066
6.5	0.2403	0.2324	0.2367	-0.0043
7.0	0.2442	0.2320	0.2426	-0.0106
8.0	0.2542	0.2510	0.2508	0.0002
9.0	0.2503	0.2523	0.2454	0.0069
10.0	0.2483	0.2409	0.2450	-0.0041
50.0	0.2500	0.2461	0.2463	-0.0002

TABLE VII

α=1.0, ω_0^2=4.0, h=10.0, c=1.0

Time	Linear	Reversion	Machine	Error
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0982	0.0982	0.0982	0.0000
1.0	0.2677	0.2675	0.2626	0.0049
1.5	0.3576	0.3554	0.3188	0.0366
2.0	0.3343	0.3211	0.2507	0.0704
2.5	0.2591	0.2173	0.1799	0.0374
3.0	0.2069	0.1230	0.1761	-0.0531
3.5	0.2064	0.0934	0.2164	-0.1230
4.0	0.2377	0.1400	0.2463	-0.1063
4.5	0.2655	0.2298	0.2410	-0.0112
5.0	0.2712	0.3026	0.2195	0.0831
5.5	0.2593	0.3085	0.2093	0.0992
6.0	0.2453	0.2451	0.2156	0.0295
6.5	0.2403	0.1619	0.2260	-0.0641
7.0	0.2442	0.1222	0.2289	-0.1067
8.0	0.2542	0.2225	0.2198	0.0027
9.0	0.2503	0.2703	0.2222	0.0481
10.0	0.2483	0.1745	0.2234	-0.0489
50.0	0.2500	0.2110	0.2225	-0.0115

$$Y(s) = \frac{\omega c}{(s^2+\omega^2)(s+A)(s+B)} - \frac{\frac{3}{4}hc^3\omega}{(s^2+\omega^2)(s+A)^4(s+B)^4} +$$

$$\frac{\frac{3}{4}hc^3\omega}{(s^2+9\omega^2)(s+A)^4(s+B)^4} . \qquad (32)$$

As in the case of the constant forcing function three solutions are considered depending on the nature of the radical in Equations (13) and (14).

1. <u>Radical Greater than Zero</u>

The inverse Laplace transform of Equation (32) is:

$$\begin{aligned} y(t) = {} & M_{11}\cos(\omega t) + M_{12}\sin(\omega t) + N_1e^{-At} + N_2e^{-Bt} + M_{13}\cos(\omega t) + \\ & M_{14}\sin(\omega t) + (N_3 + N_4t + N_5t^2 + N_6t^3)e^{-At} + \\ & (N_7 + N_8t + N_9t^2 + N_{10}t^3)e^{-Bt} + M_{15}\cos(3\omega t) + M_{16}\sin(3\omega t) + \\ & (N_{11} + N_{12}t + N_{13}t^2 + N_{14}t^3)e^{-At} + (N_{15} + N_{16}t + N_{17}t^2 + N_{18}t^3)e^{-Bt}. \end{aligned} \qquad (33)$$

The solutions represented by Equation (33) and the first four terms of Equation (33) were tested using $\alpha = 8.0$, $\omega_0^2=9.0$, $c=1.0$, $h=20.0$ and $\omega=1.0$. The solutions are compared to a machine solution in Table VIII. In Table IX the solutions are compared for the non-linear coefficient increased to fifty. Table X is a comparison of the solutons for the frequency of the forcing function increased to five and the non-linear coefficient equal to fifty. Inspection of the tables indicates that as the non-linear coefficient is

TABLE VIII

$\alpha=8.0,\ \omega_0^2=9.0,\ h=20.0,\ c=1.0,\ \omega=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.00000	0.00000	0.00000	0.00000
0.5	0.00885	0.00885	0.00885	0.00000
1.0	0.03603	0.03603	0.03603	0.00000
1.5	0.06667	0.06666	0.06661	0.00005
2.0	0.08728	0.08726	0.08704	0.00022
2.5	0.08973	0.08963	0.08923	0.00040
3.0	0.07184	0.07160	0.07128	0.00032
3.5	0.03719	0.03679	0.03674	0.00005
4.0	-0.00615	-0.00667	-0.00644	0.00023
4.5	-0.04777	-0.04833	-0.04789	0.00044
5.0	-0.07759	-0.07810	-0.07753	0.00057
5.5	-0.08835	-0.08873	-0.08803	0.00070
6.0	-0.07745	-0.07762	-0.07695	0.00067
6.5	-0.04758	-0.04750	-0.04714	0.00036
7.0	-0.00605	-0.00576	-0.00579	-0.00003
23.0	-0.01959	-0.01998	-0.01991	0.00007
24.0	-0.08311	-0.08352	-0.08298	0.00054
25.0	-0.07022	-0.07029	-0.06971	0.00058
26.0	0.00723	0.00758	0.00745	0.00013
27.0	0.07803	0.07847	0.07798	0.00049
28.0	0.07709	0.07722	0.07658	0.00064
29.0	0.00528	0.00497	0.00502	-0.00005

TABLE IX

$\alpha=8.0,\ \omega_0^2=9.0,\ h=50.0,\ c=1.0,\ \omega=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.00000	0.00000	0.00000	0.00000
0.5	0.00885	0.00885	0.00885	0.00000
1.0	0.03603	0.03603	0.03601	0.00002
1.5	0.06667	0.06665	0.06637	0.00028
2.0	0.08728	0.08715	0.08606	0.00109
2.5	0.08973	0.08922	0.08732	0.00190
3.0	0.07184	0.07064	0.06905	0.00059
3.5	0.03719	0.03519	0.03510	0.00009
4.0	-0.00615	-0.00874	-0.00731	0.00143
4.5	-0.04777	-0.05056	-0.04831	0.00225
5.0	-0.07759	-0.08015	-0.07730	0.00085
5.5	-0.08835	-0.09025	-0.08677	0.00348
6.0	-0.07745	-0.07829	-0.07508	0.00321
6.5	-0.04758	-0.04717	-0.04544	0.00173
7.0	-0.00605	-0.00458	-0.00479	-0.00021
22.8	-0.00207	-0.00375	-0.00326	0.00049
24.0	-0.08311	-0.08517	-0.08243	0.00274
25.0	-0.07022	-0.07054	-0.06780	0.00274
26.0	0.00723	0.00899	0.00831	0.00068
27.0	0.07803	0.08020	0.07769	0.00251
28.0	0.07709	0.07774	0.07465	0.00309
29.0	0.00528	0.00373	0.00402	-0.00029

TABLE X

$\alpha=8.0,\ \omega_0^2=9.0,\ h=50.0,\ c=1.0,\ \omega=5.0$

Time	Linear	Reversion	Machine	Error
0.0	0.00000	0.00000	0.00000	0.00000
0.2	0.00434	0.00434	0.00434	0.00000
0.4	0.02066	0.02066	0.02066	0.00000
0.6	0.03549	0.03549	0.03549	0.00000
0.8	0.03246	0.03246	0.03242	0.00004
1.0	0.01123	0.01121	0.01115	0.00006
1.2	-0.01136	-0.01140	-0.01144	-0.00004
1.4	-0.01662	-0.01672	-0.01670	-0.00002
1.6	-0.00136	-0.00151	-0.00141	0.00010
1.8	0.01916	0.01895	0.01913	-0.00018
2.0	0.02512	0.02487	0.02511	-0.00024
2.2	0.01032	0.01005	0.01029	-0.00024
2.4	-0.01220	-0.01248	-0.01224	0.00024
2.6	-0.02214	-0.02243	-0.02218	0.00025
2.8	-0.01069	-0.01100	-0.01072	0.00028
3.0	0.01137	0.01106	0.01137	-0.00031
6.0	0.00520	0.00515	0.00519	-0.00004
6.4	-0.02273	-0.02276	-0.02275	0.00001
6.8	0.01373	0.01370	0.01374	0.00004
7.2	0.01131	0.01129	0.01131	0.00003
7.6	-0.02314	-0.02314	-0.02316	0.00002

increased, the accuracy of the solution decreases and as the frequency of the forcing function is increased, a more accurate approximation is obtained. The effect of the magnitude of the amplitude of the forcing function was not illustrated; however from an examination of the coefficients of Equation (33) it can be seen that as c is increased it would have a more detrimental effect on the accuracy of the solution than corresponding increases of the non-linear coefficient. An examination of the solution for the non-linear coefficient equal to zero, represented by the first four terms of Equation (33), reveals that it is overall as accurate a solution as the solution utilizing all the terms of Equation (33).

2. Radical Equal to Zero

Equation (32) becomes:

$$Y(s) = \frac{\omega c}{(s^2+\omega^2)(s+A)^2} - \frac{\frac{3}{4}hc^3\omega}{(s^2+\omega^2)(s+A)^8} + \frac{\frac{3}{4}hc^3\omega}{(s^2+9\omega^2)(s+A)^8} \quad . \quad (34)$$

The inverse Laplace transform of Equation (34) yields:

$$\begin{aligned} y(t) = {} & M_{17}\cos(\omega t) + M_{18}\sin(\omega t) + (N_{19} + N_{20}t)e^{-At} + M_{19}\cos(\omega t) + \\ & M_{20}\sin(\omega t) + (N_{21}+N_{22}t+N_{23}t^2+N_{24}t^3+N_{25}t^4+N_{26}t^5+N_{27}t^6+N_{28}t^7)e^{-At} + \\ & M_{21}\cos(3\omega t)+M_{22}\sin(3\omega t)+(N_{29}+N_{30}t+N_{31}t^2+N_{32}t^3+N_{33}t^4 + \\ & N_{34}t^5 + N_{35}t^6 + N_{36}t^7)e^{-At} \quad . \qquad (35) \end{aligned}$$

Table XI is a comparison of the solutions for the parameter values $\alpha=6.0$, $\omega_0^2=9.0$, $h=20.0$, $c=1.0$ and $\omega=1.0$. A comparison of errors listed in Table XI with the errors listed in Table VIII indicates that a decrease in the magnitude of the damping coefficient results in a less accurate approximate solution. The linear solution represented by the first three terms of Equation (35) is again seen to be overall as accurate an approximation when compared to the solution represented by Equation (35).

3. <u>Radical Imaginary</u>

Utilizing the relationships expressed in Equations (25) and (26) and combining terms with complex conjugate coefficients, the inverse Laplace transform of Equation (32) becomes:

$$\begin{aligned} y(t) = {} & M_{23}\cos(\omega t) + M_{24}\sin(\omega t) + N_{37}e^{-\frac{\alpha}{2}t}\cos(Dt) + N_{38}e^{-\frac{\alpha}{2}t}\sin(Dt) + \\ & M_{25}\cos(\omega t) + M_{26}\sin(\omega t) + (N_{39}+N_{40}t+N_{41}t^2+N_{42}t^3)e^{-\frac{\alpha}{2}t}\cos(Dt) + \\ & (N_{43}+N_{44}t+N_{45}t^2+N_{46}t^3)e^{-\frac{\alpha}{2}t}\sin(Dt) + M_{27}\cos(3\omega t) + \\ & M_{28}\sin(3\omega t) + (N_{47}+N_{48}t+N_{49}t^2+N_{50}t^3)e^{-\frac{\alpha}{2}t}\cos(Dt) + \\ & (N_{51}+N_{52}t+N_{53}t^2+N_{54}t^3)e^{-\frac{\alpha}{2}t}\sin(Dt) \ . \end{aligned} \tag{36}$$

Using the values $\alpha=1.0$, $\omega_0^2=9.0$, $h=20.0$, $c=1.0$ and $\omega=1.0$, the solutions are compared in Table XII. A comparison of Table XII with Tables VIII and XI further illustrates the critical effect of the damping coefficient on the accuracy of the solution. In the three tables the parameter values are

TABLE XI

$\alpha=6.0,\ \omega_0^2=9.0,\ h=20.0,\ c=1.0,\ \omega=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.00000	0.00000	0.00000	0.00000
0.5	0.01024	0.01024	0.01025	0.00001
1.0	0.04287	0.04287	0.04285	0.00002
1.5	0.07789	0.07788	0.07766	0.00022
2.0	0.09836	0.09825	0.09746	0.00079
2.5	0.09612	0.09572	0.09449	0.00123
3.0	0.07073	0.06984	0.06912	0.00072
3.5	0.02814	0.02675	0.02714	-0.00039
4.0	-0.02132	-0.02290	-0.02171	0.00119
4.5	-0.06555	-0.06695	-0.06560	0.00135
5.0	-0.09373	-0.09470	-0.09320	0.00140
5.5	-0.09896	-0.09932	-0.09759	0.00173
6.0	-0.07996	-0.07958	-0.07827	0.00131
6.5	-0.04139	-0.04030	-0.04020	0.00010
7.0	0.00732	0.00880	0.00783	0.00097
21.0	0.09980	0.10042	0.09875	0.00167
22.0	0.05929	0.05847	0.05782	0.00065
23.0	-0.03573	-0.03723	-0.03600	0.00123
24.0	-0.09790	-0.09869	-0.09711	0.00158
25.0	-0.07006	-0.06945	-0.06846	0.00099
26.0	0.02219	0.02370	0.02256	0.00114
27.0	0.09404	0.09498	0.09349	0.00149

TABLE XII

$\alpha=1.0,\ \omega_0^2=9.0,\ h=20.0,\ c=1.0,\ \omega=1.0$

Time	Linear	Reversion	Machine	Error
0.0	0.00000	0.00000	0.00000	0.00000
0.5	0.01635	0.01635	0.01635	0.00000
1.0	0.08176	0.08174	0.08161	0.00013
1.5	0.13744	0.13703	0.13485	0.00218
2.0	0.12875	0.12637	0.12134	0.00503
2.5	0.07792	0.07206	0.07300	0.00094
3.0	0.02512	0.01893	0.02743	-0.00850
3.5	-0.02494	-0.02464	-0.02176	0.00288
4.0	-0.07801	-0.07121	-0.07877	-0.00756
4.5	-0.11866	-0.11433	-0.11872	-0.00439
5.0	-0.12570	-0.12773	-0.12008	0.00765
5.5	-0.09724	-0.09750	-0.09118	0.00632
6.0	-0.04708	-0.04019	-0.04717	-0.00698
6.5	0.01145	0.01727	0.00821	0.00906
7.0	0.06800	0.06440	0.06752	-0.00312
22.0	0.01429	0.01258	0.01643	-0.00385
23.0	-0.09595	-0.09420	-0.09641	-0.00221
24.0	-0.11798	-0.11444	-0.11215	0.00229
25.0	-0.03154	-0.02945	-0.03276	-0.00331
26.0	0.08390	0.08262	0.08404	-0.00142
27.0	0.12220	0.11872	0.11709	0.00163
28.0	0.04815	0.04568	0.04806	-0.00238

equal except for the magnitude of the damping coefficient. As α was decreased from eight in Table VIII to six in Table XI and finally to one in Table XII, the magnitude of the error increased correspondingly. The linear solution illustrated in Table XII is seen to be an overall better approximation than the reversion solution.

VI. MODIFIED SOLUTION

The reversion method previously examined was limited in application to a region in which the solution did not diverge. The reversion method can be made to always converge by modifying the solution to be correct in the steady state for any parameter values. The steady state for a constant forcing function can be obtained by solving the cubic equation:

$$\omega_0^2 y + hy^3 = c \ . \tag{37}$$

Consider a truncated form of Equation (23):

$$y(t) = K_{20} + K_{21}e^{-At} + K_{22}e^{-Bt} + K_{23} + (K_{24} + K_{25}t)e^{-At} + (K_{28} + K_{29}t)e^{-Bt} \ . \tag{38}$$

Since the coefficients K_{23} to K_{29} have the common factor hc^3, see Appendix, Equation (38) can be rewritten as:

$$y(t) = K_{20} + K_{21}e^{-At} + K_{22}e^{-Bt} - \frac{hc^3}{(\omega_0^2)^4} + hc^3(K_{24}' + K_{25}'t)e^{-At} + hc^3(K_{28}' + K_{29}'t)e^{-Bt} \ . \tag{39}$$

Let SS be equal to the steady state obtained from the solution of Equation (37). If the steady-state portion of Equation (39) is set equal to SS, then:

$$SS = K_{20} - \frac{hc^3}{(\omega_0^2)^4} \ . \tag{40}$$

Equation (40) can be rewritten as:

$$hc^3 = (\omega_0^2)^4(K_{20} - SS) \tag{41}$$

Substitution of Equation (41) into Equation (39) yields:

$$y(t) = SS + K_{21}e^{-At} + K_{22}e^{-Bt} + (\omega_0^2)^4(K_{20} - SS)(K_{24}' + K_{25}'t)e^{-At} + (\omega_0^2)^4(K_{20} - SS)(K_{28}' + K_{29}'t)e^{-Bt} . \tag{42}$$

Equation (42) is the modified solution. Using the values $\alpha=6.0$, $\omega_0^2=4.0$, $h=1.0$ and $c=1.0$, the modified solution is compared to a machine solution in Table XIII. The results indicate a relatively accurate approximate solution was obtained. In Tables XIV and XV the solutions are compared for the non-linear coefficient respectively equal to ten and fifty. The tables reveal that the error increased as h was increased; however, the modified solution is seen to be still usable as an approximation for the non-linear coefficient equal to fifty. In Table XVI the solutions are compared for the forcing function equal to ten. The magnitude of the forcing function is seen to be a more critical factor on the accuracy of the approximation than corresponding increases of the non-linear coefficient. A comparison of the modified solution with the reversion solution for similar parameter values indicates that the modified solution is a more accurate approximation for large values of h and c. For small values of h and c, the modified and reversions solutions are of the same relative accuracy, as revealed by a comparison of Table XIII with Table IV.

TABLE XIII

$\alpha=6.0,\ \omega_0^2=4.0,\ h=1.0,\ c=1.0$

Time	Modified	Machine	Error
0.0	0.0000	0.0000	0.0000
0.5	0.0532	0.0533	-0.0001
1.0	0.1133	0.1138	-0.0005
1.5	0.1559	0.1568	-0.0011
2.0	0.1849	0.1860	-0.0011
2.5	0.2046	0.2058	-0.0012
3.0	0.2180	0.2191	-0.0011
3.5	0.2271	0.2281	-0.0010
4.0	0.2333	0.2341	-0.0008
4.5	0.2374	0.2381	-0.0007
5.0	0.2403	0.2408	-0.0005
5.5	0.2422	0.2426	-0.0004
6.0	0.2435	0.2438	-0.0003
6.5	0.2444	0.2446	-0.0002
7.0	0.2450	0.2452	-0.0002
8.0	0.2457	0.2459	-0.0002
9.0	0.2460	0.2461	-0.0001
10.0	0.2461	0.2462	-0.0001
12.0	0.2462	0.2463	-0.0001
14.0	0.2463	0.2463	0.0000

TABLE XIV

$\alpha=6.0,\ \omega_0^2=4.0,\ h=10.0,\ c=1.0$

Time	Modified	Machine	Error
0.0	0.0000	0.0000	0.0000
0.5	0.0522	0.0533	-0.0011
1.0	0.1098	0.1136	-0.0038
1.5	0.1493	0.1553	-0.0060
2.0	0.1751	0.1821	-0.0070
2.5	0.1919	0.1986	-0.0067
3.0	0.2028	0.2086	-0.0058
3.5	0.2099	0.2145	-0.0046
4.0	0.2145	0.2179	-0.0034
4.5	0.2174	0.2196	-0.0022
5.0	0.2193	0.2210	-0.0017
5.5	0.2205	0.2216	-0.0011
6.0	0.2212	0.2220	-0.0008
6.5	0.2217	0.2222	-0.0005
7.0	0.2220	0.2223	-0.0003
7.5	0.2222	0.2224	-0.0002
8.0	0.2223	0.2224	-0.0001
9.0	0.2224	0.2225	-0.0001
10.0	0.2225	0.2225	0.0000

TABLE XV

$\alpha=6.0,\ \omega_0^2=4.0,\ h=50.0,\ c=1.0$

Time	Modified	Machine	Error
0.0	0.0000	0.0000	0.0000
0.5	0.0503	0.0533	-0.0030
1.0	0.1034	0.1123	-0.0089
1.5	0.1370	0.1489	-0.0119
2.0	0.1570	0.1671	-0.0101
2.5	0.1685	0.1747	-0.0062
3.0	0.1749	0.1774	-0.0025
3.5	0.1782	0.1783	-0.0001
4.0	0.1799	0.1786	0.0013
4.5	0.1805	0.1787	0.0018
5.0	0.1806	0.1787	0.0019
5.5	0.1805	0.1787	0.0018
6.0	0.1802	0.1787	0.0015
8.0	0.1793	0.1787	0.0006
10.0	0.1789	0.1787	0.0002
12.0	0.1787	0.1787	0.0000

TABLE XVI

$\alpha=6.0,\ \omega_0^2=4.0,\ h=1.0,\ c=10.0$

Time	Modified	Machine	Error
0.0	0.0000	0.0000	0.0000
0.5	0.4929	0.5326	-0.0397
1.0	1.0000	1.1077	-0.1077
1.5	1.3061	1.4187	-0.1126
2.0	1.4749	1.5294	-0.0545
2.5	1.5619	1.5545	0.0074
3.0	1.6019	1.5574	0.0445
3.5	1.6160	1.5571	0.0589
4.0	1.6166	1.5568	0.0598
4.5	1.6093	1.5568	0.0525
5.0	1.6027	1.5568	0.0459
6.0	1.5866	1.5568	0.0298
7.0	1.5746	1.5568	0.0178
8.0	1.5669	1.5568	0.0101
10.0	1.5598	1.5568	0.0030
12.0	1.5576	1.5568	0.0008
14.0	1.5570	1.5568	0.0002
16.0	1.5568	1.5568	0.0000

VII. CONCLUSIONS

A method for obtaining an approximate solution for a specific non-linear differential equation has been derived utilizing reversion-of-series and non-linear transform techniques. The usefulness of the solution was found to be dependent upon the nature of the forcing function and the parameter values specified. Also, it was found that useful approximation could be obtained for particular parameter values by simply considering as a solution that portion of the reversion approximation which was independent of the non-linear coefficient. For the case of the constant forcing function the linear approximation was found to be less accurate than the reversion solution. In the case of the sinusoidal forcing function the linear approximation was found to be as accurate as the reversion solution when the damping was significant and a more accurate approximation when the damping coefficient was relatively small.

For a constant forcing function it was found that the steady state could be specified and utilized to modify the reversion solution. The modified solution was found to be useful as an approximation over a relatively wide range of parameter values and gave results that were superior to the linear and reversion approximations.

The reversion method was used to solve a particular non-linear differential equation; however, the development of the

solution indicates the possibility of applying this method to solve non-linear differential equations similar in form to the equation considered.

APPENDIX

COEFFICIENT FORMULAS

$$K_1 = \frac{c}{A^2}$$

$$K_2 = -\frac{c}{A}$$

$$K_3 = -\frac{c}{A^2}$$

$$K_4 = -\frac{hc^3}{A^8} + \frac{3h^2c^5}{A^{14}}$$

$$K_5 = -K_4$$

$$K_n = \frac{1}{(n-5)!}\left[\frac{hc^3}{A^{(13-n)}} - \frac{3h^2c^5}{A^{(19-n)}}\right] \qquad n = 6,7,8,9,10,11,12$$

$$K_n = \frac{1}{(n-5)!}\left[-\frac{3h^2c^5}{A^{(19-n)}}\right] \qquad n = 13,14,15,16,17,18$$

$$K_{20} = \frac{c}{AB}$$

$$K_{21} = \frac{c}{A(A-B)}$$

$$K_{22} = \frac{c}{B(B-A)}$$

$$K_{23} = -\frac{hc^3}{A^4B^4}$$

$$K_{24} = hc^3\left[\frac{A(2B - 10A)(B-A) + (15A^2-6BA + B^2)(B - 3A)}{A^4(B-A)^7}\right]$$

$$K_{25} = hc^3\left[\frac{15A^2 - 6AB + B^2}{A^3(B - A)^6}\right]$$

$$K_{26} = \frac{hc^3}{2}\left[\frac{B - 5A}{A^2(B - A)^5}\right]$$

$$K_{27} = \frac{hc^3}{6}\left[\frac{1}{A(B - A)^4}\right]$$

In coefficients K_{24} to K_{27}, if A is substituted for B and B is substituted for A, then:

$$K_n = K_{n-4} \qquad n = 28, 29, 30, 31$$

$$K_{40} = \frac{c}{AB}$$

$$K_{41} = -\frac{D^2c + jDA_1c}{2D^4 + 2A_1^2D^2}$$

$$K_{43} = -\frac{hc^3}{A^4B^4}$$

$$K_{44} = \frac{hc^3}{32D^7}\left[\frac{F_1F_2 + F_3F_4 + j(F_3F_2 - F_1F_4)}{F_2^2 + F_4^2}\right]$$

$$K_{45} = \frac{hc^3}{128D^6}\left[\frac{F_5F_6 + F_7F_8 + j(F_7F_6 - F_5F_8)}{F_6^2 + F_8^2}\right]$$

$$K_{46} = \frac{hc^3}{32D^5}\left[\frac{F_9 - jF_{10}}{F_{11}}\right]$$

$$K_{47} = \frac{hc^3}{96D^4}\left[\frac{A_1 - jD}{A_1^2 + D^2}\right]$$

where

$$A_1 = \frac{\alpha}{2}$$

$$F_1 = 26A_1D^2 - 8A_1^3 + 3\omega_0^2A_1$$

$$F_2 = 4A_1D^3 - 4A_1^3D$$

$$F_3 = -26A_1^2D + 10D^3 + 6\omega_0^2D$$

$$F_4 = A_1^4 - 6A_1^2D^2 + D^4$$

$$F_5 = 32A_1^2 - 32D^2 - 12\omega_0^2$$

$$F_6 = A_1^3 - 3A_1D^2$$

$$F_7 = 56A_1D$$

$$F_8 = 3A_1^2D - D^3$$

$$F_9 = -4A_1^2D + 3D(A_1^2 - D^2)$$

$$F_{10} = 6A_1D^2 + 2A_1^3 - 2A_1D^2$$

$$F_{11} = 4A_1^2D^2 + (A_1^2 - D^2)^2$$

The coefficients K_{42}, K_{48}, K_{49}, K_{50} and K_{51} are respectively the complex conjugates of K_{41}, K_{44}, K_{45}, K_{46} and K_{47}.

$$M_1 = \mathrm{Re}(K_{41})$$

$$M_2 = \mathrm{Im}(K_{41})$$

$$M_{n+1} = \mathrm{Re}(K_{43+\frac{1}{2}n}) \qquad n = 2,\ 4,\ 6,\ 8$$

$$M_n = \mathrm{Im}(K_{42+\frac{1}{2}n}) \qquad n = 4, 6, 8, 10$$

$$M_{11} = \frac{2cR_1}{R_1^2 + R_2^2}$$

$$M_{12} = \frac{-2cR_2}{R_1^2 + R_2^2}$$

$$M_{13} = \frac{16E_1R_4}{\omega(R_4^2 + R_3^2)}$$

$$M_{14} = \frac{16E_1R_3}{\omega(R_4^2 + R_3^2)}$$

$$N_1 = \frac{c\omega}{R_5R_6}$$

$$N_2 = \frac{c\omega}{R_7R_8}$$

$$N_3 = \frac{-E_1(R_5R_6R_{18} - R_{22}R_{23})}{3R_5^4R_6^7}$$

$$N_4 = \frac{-E_1(R_{11}R_5R_6 - R_9R_{12})}{R_5^3R_6^6}$$

$$N_5 = -\frac{E_1R_9}{R_5^2R_6^5}$$

$$N_6 = \frac{E_1}{6R_5R_6^4}$$

$$N_7 = \frac{-E_1(R_7R_8R_{28} - R_{32}R_{33})}{3R_7^4R_8^7}$$

$$N_8 = \frac{-E_1(R_{13}R_7R_8 - R_{10}R_{14})}{R_7^3R_8^6}$$

$$N_9 = \frac{-E_1R_{10}}{R_7^2R_8^5}$$

$$N_{10} = \frac{E_1}{6R_7R_8^4}$$

where

$$E_1 = -\frac{3}{4}\,hc^3\omega$$

$$R_1 = -2\omega(A + B)$$

$$R_2 = -2(AB - \omega^2)$$

$$R_3 = R_2^4 + R_1^4 - 6R_2^2R_1^2$$

$$R_4 = 4R_2R_1^3 - 4R_2^3R_1$$

$$R_5 = A^2 + \omega^2$$

$$R_6 = B - A$$

$$R_7 = B^2 + \omega^2$$

$$R_8 = A - B$$

$$R_9 = 3A^2 - AB + 2\omega^2$$

$$R_{10} = 3B^2 - AB + 2\omega^2$$

$$R_{11} = B - 6A$$

$$R_{12} = 9A^2 - 4AB + 5\omega^2$$

$$R_{13} = A - 6B$$

$$R_{14} = 9B^2 - 4AB + 5\omega^2$$

$$R_{15} = 54\omega^2 + 6B^2$$

$$R_{16} = 84A^3 - 42BA^2$$

$$R_{17} = R_{15}A - 6B\omega^2$$

$$R_{18} = R_{16} + R_{17}$$

$$R_{19} = - 21\ A^4 + 14\ BA^3$$

$$R_{20} = - A^2(27\omega^2 + 3B^2)$$

$$R_{21} = 6BA\omega^2 - 10\omega^4 + B^2\omega^2$$

$$R_{22} = R_{19} + R_{20} + R_{21}$$

$$R_{23} = 12A^2 - 6AB + 6\omega^2$$

$$R_{25} = 54\omega^2 + 6A^2$$

$$R_{26} = 84B^3 - 42AB^2$$

$$R_{27} = R_{25}B - 6A\omega^2$$

$$R_{28} = R_{26} + R_{27}$$

$$R_{29} = - 21B^4 + 14AB^3$$

$$R_{30} = - (27\omega^2 + 3A^2)B^2$$

$$R_{31} = 6AB\omega^2 - 10\omega^4 + A^2\omega^2$$

$$R_{32} = R_{29} + R_{30} + R_{31}$$

$$R_{33} = 12B^2 - 6AB + 6\omega^2$$

In coefficients M_{13}, M_{14}, N_3 to N_{10}, if $-E_1$ is substituted for E_1 and 3ω is substituted for ω, the coefficients M_{15}, M_{16}, N_{11} to N_{18} can be obtained.

$$M_{17} = -\frac{2A\omega}{R_{34}}$$

$$M_{18} = \frac{(A^2 - \omega^2)}{R_{34}}$$

$$M_{19} = \frac{-2E_1 R_{38}}{R_{38}^2 + R_{37}^2}$$

$$M_{20} = \frac{-2E_1 R_{37}}{R_{38}^2 + R_{37}^2}$$

$$N_{19} = \frac{2A\omega c}{(A^2 + \omega^2)^2}$$

$$N_{20} = \frac{\omega c}{A^2 + \omega^2}$$

$$N_{21} = \frac{E_1}{21}\left[\frac{(A^2+\omega^2)R_{40} + 14AR_{39}}{(A^2 + \omega^2)^8}\right]$$

$$N_{22} = \frac{E_1 R_{39}}{3(A^2 + \omega^2)^7}$$

$$N_{23} = \frac{E_1 R_{41}}{(A^2 + \omega^2)^6}$$

$$N_{24} = \frac{E_1 R_{42}}{6(A^2 + \omega^2)^5}$$

$$N_{25} = \frac{E_1(A^2 - A\omega^2)}{6(A^2 + \omega^2)^4}$$

$$N_{26} = \frac{E_1}{120}\left[\frac{3A^2 - \omega^2}{(A^2 + \omega^2)^3}\right]$$

$$N_{27} = \frac{AE_1}{360}\left[\frac{1}{(A^2 + \omega^2)^2}\right]$$

$$N_{28} = \frac{E_1}{5040(A^2 + \omega^2)}$$

where

$$E_1 = -\frac{3}{4}hc^3\omega$$

$$R_{34} = 4A^2\omega^2 + (A^2 - \omega^2)^2$$

$$R_{35} = A^4 + \omega^4 - 6A^2\omega^2$$

$$R_{36} = 4A\omega^3 - 4A^3\omega$$

$$R_{37} = -2\omega(R_{35}^2 - R_{36}^2)$$

$$R_{38} = -4\omega R_{35}R_{36}$$

$$R_{39} = 21A^6 - 105\omega^2A^4 + 63\omega^4A^2 - 3\omega^6$$

$$R_{40} = -126A^5 + 420\omega^2A^3 - 126\omega^4A$$

$$R_{41} = 3A^5 - 10\omega^2A^3 + 3\omega^4A$$

$$R_{42} = 5A^4 - 10\omega^2A^2 + \omega^4$$

In coefficients M_{19}, M_{20}, N_{21} to N_{28}, if $-E_1$ is substituted for E_1 and 3ω is substituted for ω, the coefficients M_{21}, M_{22}, N_{29} to N_{36} can be obtained.

$$M_{23} = \frac{2cL_{13}}{L_{13}^2 + L_{14}^2}$$

$$M_{24} = \frac{-2cL_{14}}{L_{13}^2 + L_{14}^2}$$

$$M_{25} = \frac{16E_1L_{18}}{\omega(L_{18}^2 + L_{17}^2)}$$

$$M_{26} = \frac{16E_1L_{17}}{\omega(L_{18}^2 + L_{17}^2)}$$

$$N_{37} = \frac{2c\omega L_{15}}{L_{15}^2 + L_{16}^2}$$

$$N_{38} = \frac{-2c\omega L_{16}}{L_{15}^2 + L_{16}^2}$$

$$N_{39} = \frac{4L_{61}L_{63}}{L_{62}}$$

$$N_{40} = \frac{2L_{32}}{L_{31}}$$

$$N_{41} = \frac{L_{20}}{L_{22}} (- L_4L_{21} + 3L_2L_3)$$

$$N_{42} = \frac{L_{19}L_1}{3(L_1^2 + L_2^2)}$$

$$N_{43} = \frac{4L_{61}L_{64}}{L_{62}}$$

$$N_{44} = \frac{2L_{33}}{L_{31}}$$

$$N_{45} = - \frac{L_{20}}{L_{22}} (L_3L_{21} + 3L_2L_4)$$

$$N_{46} = \frac{-L_{19}L_2}{3(L_1^2 + L_2^2)}$$

where

$$A_1 = \frac{\alpha}{2}$$

$$E_1 = -\frac{3}{4}\,hc^3\omega$$

$$L_1 = A_1^2 - D^2 + \omega^2$$

$$L_2 = 2DA_1$$

$$L_3 = L_1^2 - L_2^2$$

$$L_4 = 2L_1L_2$$

$$L_5 = L_1L_3 - L_2L_4$$

$$L_6 = L_1L_4 + L_2L_3$$

$$L_7 = L_1L_5 - L_2L_6$$

$$L_8 = L_1L_6 + L_2L_5$$

$$L_9 = A_1^2 - D^2$$

$$L_{10} = A_1L_9 - DL_2$$

$$L_{11} = A_1L_2 + DL_9$$

$$L_{12} = A_1^2 + D^2$$

$$L_{13} = -4A_1\omega$$

$$L_{14} = -2(L_{12} - \omega^2)$$

$$L_{15} = 2DL_2$$

$$L_{16} = -2DL_1$$

$$L_{17} = L_{14}^4 + L_{13}^4 - 6L_{14}^2 L_{13}^2$$

$$L_{18} = 4L_{14}L_{13}^3 - 4L_{14}^3 L_{13}$$

$$L_{19} = \frac{E_1}{16D^4}$$

$$L_{20} = \frac{L_{19}}{D}$$

$$L_{21} = 3L_9 - L_{12} + 2\omega^2$$

$$L_{22} = L_4^2 + L_3^2$$

$$L_{23} = \frac{E_1}{64D^6}$$

$$L_{24} = 9L_9 - 4L_{12} + 5\omega^2$$

$$L_{25} = -5A_1L_1 + 7DL_2$$

$$L_{26} = -7DL_1 + 5A_1L_2$$

$$L_{27} = L_{21}L_{24} - 27L_2^2$$

$$L_{28} = 3L_2L_{24} + 9L_2L_{21}$$

$$L_{29} = L_{23}(2DL_{26} - L_{27})$$

$$L_{30} = -L_{23}(2DL_{25} + L_{28})$$

$$L_{31} = L_5^2 + L_6^2$$

$$L_{32} = L_5L_{29} + L_6L_{30}$$

$$L_{33} = L_5L_{30} - L_6L_{29}$$

$$L_{34} = 84L_{10} - 42(A_1L_9 + DL_2)$$

$$L_{35} = 84L_{11} - 42(A_1L_2 - DL_9)$$

$$L_{36} = 6L_9 + 54\omega^2$$

$$L_{37} = A_1L_{36} + 6DL_2$$

$$L_{38} = DL_{36} - 6A_1L_2$$

$$L_{39} = - 6\omega^2A_1$$

$$L_{40} = 6\omega^2D$$

$$L_{41} = L_{34} + L_{37} + L_{39}$$

$$L_{42} = L_{35} + L_{38} + L_{40}$$

$$L_{43} = A_1L_{10} - DL_{11}$$

$$L_{44} = A_1L_{11} + DL_{10}$$

$$L_{45} = A_1L_{10} + DL_{11}$$

$$L_{46} = A_1L_{11} - DL_{10}$$

$$L_{47} = 14L_{45} - 21L_{43}$$

$$L_{48} = 14L_{46} - 21L_{44}$$

$$L_{49} = 27\omega^2 + 3L_9$$

$$L_{50} = - L_9L_{49} - 3L_2^2$$

$$L_{51} = - L_2L_{49} + 3L_2L_9$$

$$L_{52} = 6\omega^2 L_{12} - 10\omega^4 + \omega^2 L_9$$

$$L_{53} = - \omega^2 L_2$$

$$L_{54} = L_{47} + L_{50} + L_{52}$$

$$L_{55} = L_{48} + L_{51} + L_{53}$$

$$L_{56} = 6\omega^2 - 6L_{12} + 12L_9$$

$$L_{57} = 2D(L_1 L_{41} - L_2 L_{42})$$

$$L_{58} = 2D(L_2 L_{41} + L_1 L_{42})$$

$$L_{59} = L_{54} L_{56} - 12L_2 L_{55}$$

$$L_{60} = L_{55} L_{56} + 12L_2 L_{54}$$

$$L_{61} = \frac{-E_1}{768D^7}$$

$$L_{62} = L_8^2 + L_7^2$$

$$L_{63} = - L_8 L_{58} + L_8 L_{59} - L_7 L_{57} - L_7 L_{60}$$

$$L_{64} = L_8 L_{57} + L_8 L_{60} - L_7 L_{58} + L_7 L_{59}$$

In coefficients M_{25}, M_{26}, N_{39} to N_{46}, if $-E_1$ is subsitituted for E_1 and 3ω is substituted for ω, the coefficients M_{27}, M_{28}, N_{47} to N_{54} can be obtained.

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13. ABSTRACT

The reversion-of-series method is extended to the s - domain by utilizing non-linear Laplace transforms. The reversion of series in the s - domain is applied to a non-linear differential equation and approximate solutions are obtained. The approximate solution is modified for the case where the steady state is a constant value by calculating the exact steady-state value and applying it to the reversion approximation. The non-linear differential equation considered is Duffing's equation with a damping term and sinusoidal and constant forcing functions. The theoretical solutions are compared to machine solutions.

14 KEY WORDS	LINK A		LINK B		LINK C	
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